CGAL's Arrangement Package and Derivatives

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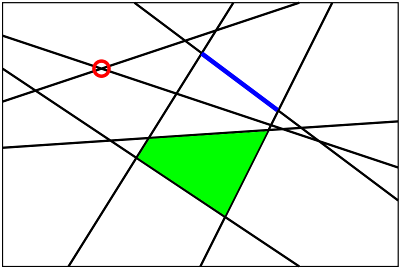
Outline

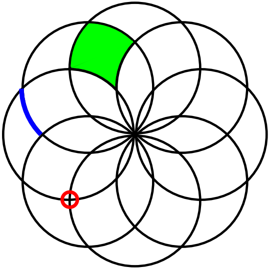
1. Arrangements
2. Boolean Set Operations
3. Envelopes
4. Minkowski Sums

Arrangement Definition

Given a collection Γ of planar curves, the arrangement $\mathcal{A}(\Gamma)$ is the partition of the plane into vertices, edges and faces induced by the curves of Γ



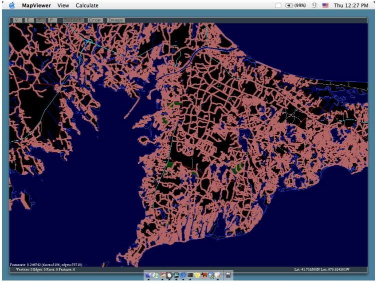
An arrangement of lines

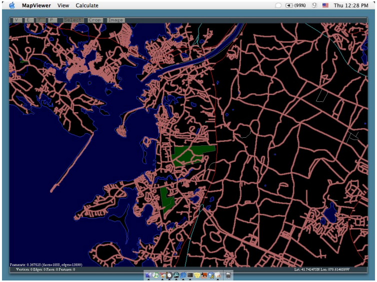


An arrangement of circles

Arrangement Background

- Arrangements have numerous applications
  - robot motion planning, computer vision, GIS, optimization, computational molecular biology





A planar map of the Boston area showing the top of the arm of cape cod.

Raw data comes from the US Census 2000 TIGER/line data files.

The `Arrangement_2` Package

- Constructs, maintains, modifies, traverses, queries, and presents subdivisions of the plane
- Robust and exact
  - All inputs are handled correctly (including degenerate)
  - Exact number types are used to achieve exact results
- Efficient
- Generic, easy to interface, extend, and adapt
- Part of the CGAL basic library

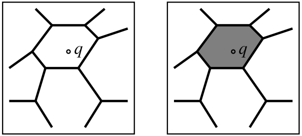
Arrangement Traits Classes

- Define the family of curves
- Aggregate geometric types and operations over the types

Curve Family	Arithmetic	Boundness	Class Name
linear segments	rational	bounded	Arr_non_caching_segment_traits_2 Arr_segment_traits_2
linear segments, rays, lines	rational	bounded	Arr_linear_traits_2
piecewise linear curves	rational	bounded	Arr_polyline_traits_2
circular arcs, linear segments	rational	bounded	Arr_circle_segment_traits_2 Arr_circular_line_arc_traits_2
algebraic curves degree ≤ 2	algebraic	bounded unbounded	Arr_conic_traits_2 Arr_conix_traits_2
algebraic curves degree ≤ 3	algebraic	unbounded	Arr_cubix_traits_2
algebraic curves degree ≤ 4	algebraic	unbounded	Arr_quadrix_traits_2
planar Bézier curves	algebraic	unbounded	Arr_Bezier_curve_traits_2
rational function arcs	algebraic	unbounded	Arr_rational_arc_traits_2

Arrangement Point Location

Given a subdivision A of the space into cells and a query point q, find the cell of A containing q



- Fast query processing
- Reasonably fast preprocessing
- Small space data structure

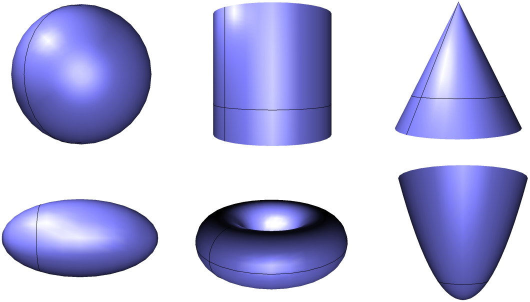
	Naive	Walk	RIC	Landmarks	Triangulat	PST
Preprocess time	none	none	$O(n \log n)$	$O(k \log k)$	$O(n \log n)$	$O(n \log n)$
Memory space	none	none	$O(n)$	$O(k)$	$O(n)$	$O(n \log n)^{(*)}$
Query time	bad	reasonable	good	good	quite good	good
Code	simple	quite simple	complicated	quite simple	modular	complicated

Walk — Walk along a line **RIC** — Random Incremental Construction based on trapezoidal decomposition
Triangulat — Triangulation **PST** — Persistent Search Tree
k — number of landmarks
(*) Can be reduced to $O(n)$

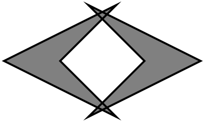
Arrangements on Surfaces — A Glimpse to the Future

A direction for future development is to extend the package to support arrangements on continuous two-dimensional parametric surfaces, which may

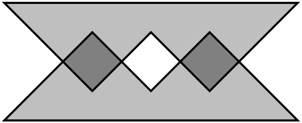
- be unbounded
- contain curves of discontinuity
- contain singularity points



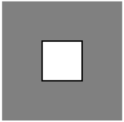
Boolean Set Operations



Union



Intersection



Complement

For two point sets P and Q and a point r:

Complement	$R = \overline{P}$	
Union	$R = P \cup Q$	
Intersection	$R = P \cap Q$	$= \overline{\overline{P} \cup \overline{Q}}$
Difference	$R = P \setminus Q$	$= P \cap \overline{Q}$
Symmetric Difference	$R = (P \setminus Q) \cup (Q \setminus P)$	
Intersection predicate	$P \cap Q \neq \emptyset$	Overlapping cell(s) are not explicitly computed
Containment predicate	$r \in P$	
Interior, Boundary, Closure		
Regularization	$R = \text{closure}(\text{interior}(P))$	

The `Nef_polyhedron_{2,S2,3}` Packages

A Nef polyhedron is obtained from a finite set of open halfspaces by set complement and set intersection operations.

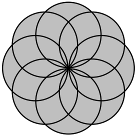
- Supports the construction of Nef polyhedron

  `Nef_polyhedron_2` — in $\mathbb{R}^2$
  `Nef_polyhedron_S2` — embedded on $\mathbb{S}^2$
  `Nef_polyhedron_3` — in $\mathbb{R}^3$
- Supports
  - Boolean set operations
  - interior, boundary, and closure operations
    - composed operations, e.g., regularization
- Robust and efficient

The `Boolean_set_operation_2` Package

- Supports
  - regularized Boolean set-operations
  - intersection predicates
  - point containment predicates
- Operands and results are regularized point sets bounded by x-monotone curves referred to as general polygons
  - General polygons may have holes
- Based on the `Arrangement_2` and `Polygon_2` packages
- Robust and efficient
  - Extremely efficient aggregated operations

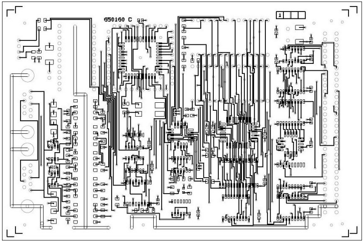
The union of eight unit discs

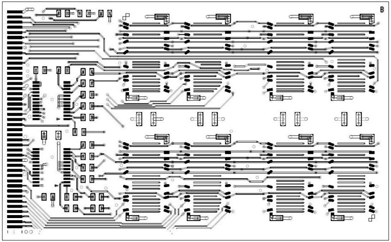


VLSI: Union of PCB Components

Input File	No. of Polygons	No. of Circles	Union Size			Time) (sec.)
			V	E	F	
VLSI_1	2593	645	13130	13130	614	3.50
VLSI_2	22406	294	14698	14698	357	24.70

V — no. of vertices **E** — no. of edges **F** — no. of faces



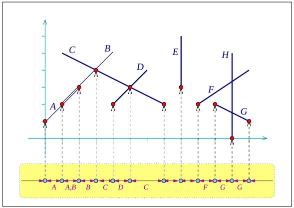


Courtesy of Maniabarco Inc.

Envelopes in $\mathbb{R}^2$

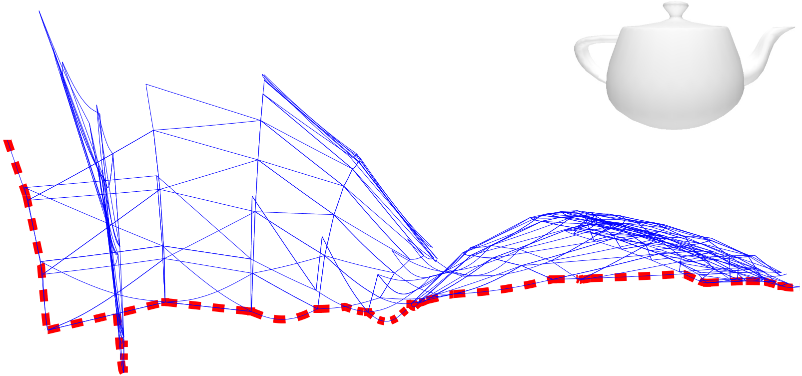
Given a set of x-monotone curves $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$, their lower envelope (resp. upper envelope) is the point-wise minimum (resp. maximum) of all curves.

The minimization diagram (resp. maximization diagram) of $\mathcal{C}$ is the subdivision of the x-axis into cells, such that the identity of the curves that induce the lower (resp. upper) envelope over a specific cell of the subdivision (an edge or a vertex) is the same.



- The minimization diagram of 8 line segments
- Each diagram vertex points to the point associated with it
- Each diagram edge is marked with segments that induce it

Computer Aided Manifacturing

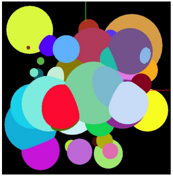
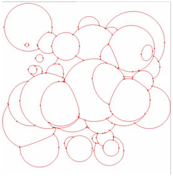
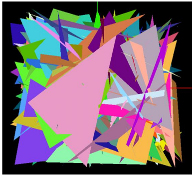
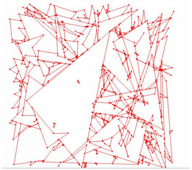


The lower envelope of a set of line segments and hyperbolic arcs obtained by the radial projection of a triangulated Utah teapot.

Envelopes in $\mathbb{R}^3$

Given a set of *xy*-monotone surfaces $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$, their lower envelope (resp. upper envelope) is the point-wise minimum (resp. maximum) of all surfaces.

The minimization diagram (resp. maximization diagram) of $\mathcal{S}$ is the subdivision of the *xy*-plane into cells, such that the identity of the surfaces that induce the lower (resp. upper) diagram over a specific cell of the subdivision (a face, an edge, or a vertex) is the same.

The `Envelope_3` Package

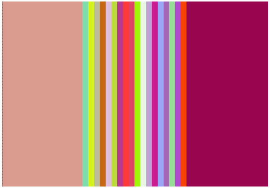
- Constructs lower and upper envelopes of surfaces

Surface Family	Class Name	
triangles	`Env_triangle_traits_3`	
spheres	`Env_sphere_traits_3`	
planes and half planes	`Env_plane_traits_3`	
quadrics	`Env_quadric_traits_3`	mpii

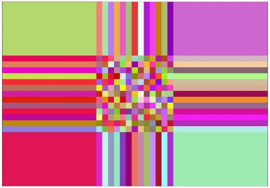
- Based on the `Arrangement_2` package
- Exploits
  - Overlay computation (using the sweep line framework)
  - Isolated points
  - Zone traversal
- Robust and efficient

Voronoi Diagrams of Points in the Plane

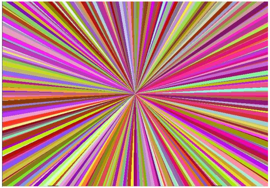
- Computed as upper envelopes of planes
- Represented as planar arrangements of unbounded curves



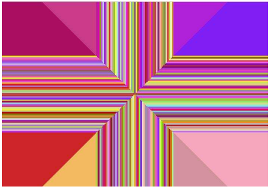
points along a line segment



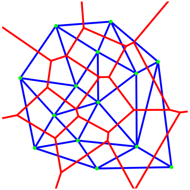
points on a grid inside a square



points on a circle



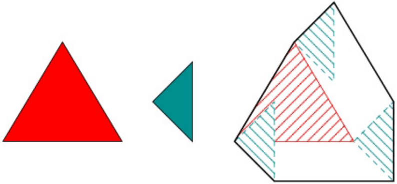
points on a square boundary



Minkowski Sum in $\mathbb{R}^2$ Background

- Given two sets P and Q in the plane, their Minkowski sum, denoted $A \oplus B$ is:

$$P \oplus Q = \{p + q \,|\, p \in P, q \in Q\}$$



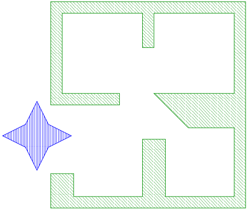
- Minkowski sums are used in many applications
  - motion planning, computer-aided design, manufacturing

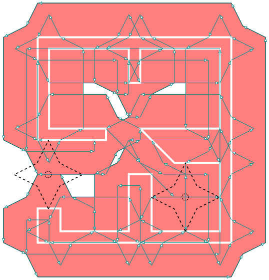
The Minkowski_sum_2 Package

- Based on the `Arrangement_2`, `Polygon_2`, and `Partition_2` packages
- Works well with the `Boolean_set_operations_2` package
  - e.g., It is possible to compute the union of offset polygons
- Robust and efficient
- Supports Minkowski sums of two simple polygons
  - Implemented using either decomposition or convolution
  - Exact
- Supports Minkowski sums of a simple polygon and a disc (polygon offseting)
  - Offers either an exact computation or a conservative approximation scheme

Motion Planning: Computing the free space

- The input robot and the obstacle are non-convex
- Exploits the convolution method
- The output sum contains four holes





For Further Reading I

D. Halperin at al.
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For Further Reading II

- M. Meyerovitch and D. Halperin.
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  Proc. 14th Annual European Symposium on Algorithms (ESA), 4168:792–803, 2006.

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  Proc. 14th Annual European Symposium on Algorithms (ESA), 4168:829-840, 2006.

- E. Berberich, E. Fogel, D. Halperin, and R. Wein
  Sweeping over Curves and Maintaining Two-Dimensional Arrangements on Surfaces.
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For Further Reading III

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  EXACUS: Efficient and exact algorithms for curves and surfaces.
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  Exact, Efficient and Complete Arrangement Computation for Cubic Curves.
  Computational Geometry, 35(1-2):36–73, Elsevier, 2006

For Further Reading IV

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